

Partition Identities From Partial Supersymmetry

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Abstract

In the quantum theory, using the notion of partial supersymmetry, in which some, but not all, operators have superpartners we derive the Euler theorem in partition theory. The parafermionic partition function gives another identity in partition theory with restrictions. Also an explicit formula for the graded parafermionic partition function is obtained. It turns out that the ratio of the former partition function to the latter is given in terms of the Jacobi Theta function, θ_4 . The inverted graded parafermionic partition function is shown to be a generating function of partitions of numbers with restriction that generalizes the Euler generating function and as a result we obtain new sequences of partitions of numbers with given restrictions.

1 INTRODUCTION

In arithmetic quantum theories the spectrum is chosen to be logarithmic in order to connect quantum mechanics both to number theory (multiplicative number theory), since the partition function is related to the Riemann zeta or other Dirichlet series [1] and to string theory [2]. In these theories number theoretic identities have been derived and interpreted. Spector in [3] introduced a notion of partial supersymmetry in which some, not all, bosonic states or operators have superpartners, with this notion again he derived and interpreted the fermionic and parafermionic thermal partition functions. In this paper we use the notion of supersymmetry in a quantum theory free of a logarithmic spectrum and connect it to the additive number theory, [4] since our partition function is related to the Euler generating function for partitions. In so doing we have derived the Euler theorem from a fermionic partition function, the theorem says that the number of partitions of a number k containing odd numbers only equals the number of partitions of a number k without duplication. Similarly, from parafermionic partition function we have derived another identity in partition theory [5] which equates the number of partitions of k in which no part appears more than $s - 1$ times, with the number of partitions of k such that no part is divisible by s . Using this identity we prove Andrews' result [5] in connection with generating functions that exclude squares and their generalizations [6]. In the graded parafermionic case we have obtained an explicit formula for the partition function and have shown that it is the inverted parafermionic partition function that corresponds to the generalized generating function for partitions in the sense of Euler since the Euler's generating function of partitions it has the interpretation of being the inverted graded fermionic partition function. With this generating function new sequences of partitions have appeared [7]

2 PARTITION FUNCTIONS AND THE EULER IDENTITY

Here we first review the concept of partial supersymmetry introduced by Spector [3], then using this concept we obtain the fermionic partition function written in terms of the bosonic partition function and the graded fermionic partition function and as a result the Euler identity is obtained. The Euler identity says that the generating functions for the number of partitions of a given number k into distinct parts and the number of partitions of k into odd parts are equal. In this paper we will be considering non-interacting quantum field theories which are not of a logarithmic spectrum, and so we can write the bosonic(fermionic) Hamiltonian without the logarithm of a prime in the form,

$$H_B = \omega \sum_{k=1}^{\infty} b_k^{\dagger} b_k. \quad (1)$$

$$H_F = \omega \sum_{k=1}^{\infty} f_k^{\dagger} f_k \quad (2)$$

where $b_k^{\dagger}(b_k)$ are the bosonic creation(annihilation) operators respectively and $f_k^{\dagger}(f_k)$ are the fermionic creation (annihilation) operators respectively. The basic idea of partial supersymmetry is that not all bosons have superpartners, therefore if we start with a bosonic partition function in which the Hamiltonian is decomposed into bosonic and fermionic parts and then using the Witten index [8] the bosons are cancelled and we are left with the fermionic partition function. Suppose we define operators q_k and c_k such that $q_k = (b_k)^2$, $q_k^{\dagger} = (b_k^{\dagger})^2$ and $(c_k)^2 = (c_k^{\dagger})^2 = 0$, note that

c_k and c_k^\dagger have the same effect as b_k and b_k^\dagger respectively the difference being that they are square free. Equivalently the bosonic Hamiltonian is

$$H_B = \omega \sum_{k=1}^{\infty} c_k^\dagger c_k + \omega \sum_{k=1}^{\infty} 2q_k^\dagger q_k. \quad (3)$$

Now, since the components of our decomposition of the Hamiltonian, H_B commute then the trace in the bosonic partition function decomposes as

$$\text{Tr} [\exp -\beta H_B] = \text{Tr} \left[\exp -\beta \omega \sum_{k=1}^{\infty} c_k^\dagger c_k \right] \text{Tr} \left[\exp -\beta \omega \sum_{k=1}^{\infty} 2q_k^\dagger q_k \right], \quad (4)$$

where the first term on the right of this equation is the fermionic partition function and the second term is the bosonic partition function. Now adding the term $\omega \sum_{k=1}^{\infty} 2f_k^\dagger f_k$ to the original Hamiltonian and then using the witten index

$$\text{Tr} [(-)^F \exp -\beta(H_B + H_F)] = 1,$$

where F is the fermion number operator with eigenvalues 0 or 1. Note that partial supersymmetry here means we have fermionic superpartners for some of the bosonic creation operators (q_k and q_k^\dagger) but not others (c_k and c_k^\dagger) so one does not grade the c_k partition function. Therefore we obtain the following factorization identity;

$$\begin{aligned} Z_f(\beta) &= \text{Tr} \left[\exp(-\beta \omega \sum_{k=1}^{\infty} c_k^\dagger c_k) \right] = \text{Tr} \left[(-1)^F \exp -\beta \omega \sum_{k=1}^{\infty} (c_k^\dagger c_k + 2q_k^\dagger q_k + 2f_k^\dagger f_k) \right] \\ &= \text{Tr} [(-)^F \exp -\beta(H_B + 2H_F)]. \end{aligned} \quad (5)$$

The bosonic and the fermionic partition functions in equation (5) are computed in the Fock space of states for both bosons and fermions using the expressions for H_B and H_F given by equation (1) and equation (2) respectively, and so we have,

$$Z_b(\beta) = \text{Tr} [\exp -\beta H_B] = \text{Tr} \left[\exp -\beta \omega \sum_{k=1}^{\infty} b_k^\dagger b_k \right] = \prod_{k=1}^{\infty} \sum_{i=0}^{\infty} x^{ik} = \prod_{k=1}^{\infty} \frac{1}{1 - x^k}, \quad (6)$$

where we have set $x = \exp -(\beta \omega)$, one can easily check that the partition function for equation (3) and that for H_B equation (1) agree. The computation in the fermionic case is similar except that due to the Fermi-Dirac statistics i takes the values 0 and 1 therefore

$$Z_f(\beta) = \text{Tr} \left[\exp -\beta \omega \sum_{k=1}^{\infty} c_k^\dagger c_k \right] = \prod_{k=1}^{\infty} \sum_{i=0}^1 x^{ik} = \prod_{k=1}^{\infty} (1 + x^k). \quad (7)$$

The remaining partition function to be computed is the graded fermionic partition

$$\text{Tr} [(-)^F \exp -\beta(2H_F)].$$

Since the fermion number (the eigenvalue of F) is either 0 or 1, then if we denote this partition by $\Delta_F(\beta)$ we have

$$\Delta_F(2\beta) = \text{Tr} \left[(-1)^F \exp(-2\beta \omega \sum_{k=1}^{\infty} c_k^\dagger c_k) \right] = \prod_{k=1}^{\infty} \sum_{i=0}^1 (-1)^i x^{ik} = \prod_{k=1}^{\infty} (1 - x^{2k}). \quad (8)$$

Finally, the factorization identity gives

$$\prod_{k=1}^{\infty} (1 + x^k) = \prod_{k=1}^{\infty} \frac{(1 - x^{2k})}{(1 - x^k)} = \prod_{k=1}^{\infty} \frac{(1 - x^{2k})}{(1 - x^{2k})(1 - x^{2k-1})} = \prod_{k=1}^{\infty} \frac{1}{(1 - x^{2k-1})}. \quad (9)$$

This is a well known theorem due to Euler [4] which says that the generating functions for the number of partitions of a given number k into distinct parts and the number of partitions of k into odd parts are equal. Here we have obtained this equality using partial supersymmetry. Next we will write down the partitions function for the bosonic and fermionic simple harmonic oscillators respectively, then connect them through partial supersymmetry and as a result we get a well known identity for $\cosh x$ in terms of an infinite product. The Hamiltonian for the bosonic(fermionic) simple harmonic oscillator are $H_B = \omega(b^\dagger b + 1/2)$ ($H_F = \omega(f^\dagger f - 1/2)$) respectively and hence their partition functions are,

$$\begin{aligned} Z_b(\beta) &= \text{Tr} \left[\exp(-\beta\omega(b^\dagger b + 1/2)) \right] \\ &= \sum_{n=0}^{\infty} \exp(-\beta\omega(n + 1/2)) = \frac{1}{2 \sinh(\beta\omega/2)}. \end{aligned} \quad (10)$$

$$\begin{aligned} Z_f(\beta) &= \text{Tr} \left[\exp(-\beta\omega(f^\dagger f - 1/2)) \right] \\ &= \sum_{n=0}^1 \exp[-\beta\omega(n - 1/2)] = 2(\cosh(\beta\omega/2)), \end{aligned} \quad (11)$$

The other piece we need to compute is the graded fermionic partition function

$$\Delta_F(2\beta) = \text{Tr} \left[(-1)^F \exp(-\beta\omega(f^\dagger f - 1/2)) \right]$$

which is simple to calculate since the eigenvalues of the fermion number operator F are 0 and 1 and so we have $\Delta_F(2\beta) = 2 \sinh(\beta\omega/2)$ since the infinite product representation of sine hyperbolic is $\sinh(x) = x \prod_{k=1}^{\infty} (1 + \frac{x^2}{k^2\pi^2})$. Then the factorization identity (5) gives

$$Z_f(\beta) = 2 \cosh(\beta\omega/2) = \frac{2 \sinh(\beta\omega)}{2 \sinh(\beta\omega/2)} = 2 \prod_{k=1}^{\infty} \frac{(1 + \frac{(\beta\omega)^2}{k^2\pi^2})}{(1 + \frac{(\beta\omega)^2}{4(k)^2\pi^2})} = 2 \prod_{k=1}^{\infty} (1 + \frac{(\beta\omega)^2}{(2k+1)^2\pi^2}), \quad (12)$$

which is exactly the infinite product representation of $(\cosh(\beta\omega)/2)$ therefore we see that the results one obtain from partial supersymmetry depends very much on the Hamiltonian used. If one is dealing with a theory with a logarithmic spectrum then partial supersymmetry gives a proof of a number theoretic identity that connects the zeta function to the modulus of Mobius inversion function [3]. Here partial supersymmetry it gives the generating function proof of the Euler theorem in which the number of partitions of k into distinct parts equals the number of partitions of k into odd parts. Also in the case the Hamiltonian is that of a harmonic oscillator we have derived a well known identity in hyperbolic trigonometry $\cosh(\beta\omega)/2 = \frac{\sinh(\beta\omega)}{\sinh(\beta\omega/2)}$. From which we obtain the infinite product representation of $\cosh x$ knowing that of $\sinh(x)$.

3 PARA-FERMIONIC PARTITION FUNCTIONS AND PARTITIONS WITH RESTRICTIONS

The natural generalization to the previous section, in which the fermionic partition function was factorized as a product of graded fermionic partition function $\Delta(2\beta)$ times the bosonic partition function $Z_b(\beta)$, would be to consider parafermions of order s . As fermions are of order 2 therefore, just like the first factorization identity (5) we will have the following second factorization identity

$$Z_s(\beta) = \text{Tr} [(-)^F \exp -\beta(H_B + sH_F)] , \quad (13)$$

where the Hamiltonian H_B is constructed out of certain operators χ_k and r_k such that $(\chi_k)^s = (\chi_k^\dagger)^s = 0$ but no lower powers vanish as operators, i.e., these are the parafermionic operators and the bosonic operators are $(r_k)^s = (b_k)^s$, $(r_k^\dagger)^s = (b_k^\dagger)^s$ thus

$$H_B = \omega \sum_{k=1}^{\infty} \chi_k^\dagger \chi_k + \omega \sum_{k=1}^{\infty} s r_k^\dagger r_k. \quad (14)$$

By the Witten index, the parafermionic partition reads,

$$Z_s(\beta) = \text{Tr} \exp(-\beta(H_s)) = \text{Tr} [(-)^F \exp(\beta(H_B + sH_F))] \quad (15)$$

where $H_s = \omega \sum_{k=1}^{\infty} \chi_k^\dagger \chi_k$. The parafermionic partition function is a sort of truncated bosonic partition since the term x^{sk} and higher terms are not present and so $Z_s(\beta) = \prod_{k=1}^{\infty} (1 + x^k + x^{2k} + \dots + x^{(s-1)k})$ and therefore, using the second factorization identity, we obtain

$$\begin{aligned} Z_s(\beta) &= \prod_{k=1}^{\infty} (1 + x^k + x^{2k} + \dots + x^{(s-1)k}) \\ &= \prod_{k=1}^{\infty} \frac{(1 - x^{sk})}{(1 - x^k)}. \end{aligned} \quad (16)$$

This is exactly an identity in the theory of partitions [5] which says that the generating function for the number of partitions of k in which no parts occur more than $s-1$ times equals the generating function for the number of partitions of k such that no parts is divisible by s . This is understood as we have eliminated terms of the form $\prod_{k=1}^{\infty} (1 - x^{sk})$ from the bosonic partition function which in turn is the generating function for the number of partitions of k without restrictions. Therefore this well known result in partition theory with restriction is obtained from partial supersymmetry and our Hamiltonians H_s , H_B and H_F . Before proving some theorems in the theory of partitions using the second factorization identity, let us first consider the following simple example, take a parafermion of order three so its partition function by the second factorization identity is $Z_3(\beta) = \prod_{k=1}^{\infty} (1 + x^k + x^{2k}) = \prod_{k=1}^{\infty} \frac{(1 - x^{3k})}{(1 - x^k)}$, the right hand side of this identity can be simplified to give

$$Z_3(\beta) = \prod_{k=1}^{\infty} \frac{1}{(1 - x^{3k-2})} \frac{1}{(1 - x^{3k-1})}, \quad (17)$$

where we have used the identity $\prod_{k=1}^{\infty} (1 - x^{3k-2})(1 - x^{3k-1})(1 - x^{3k}) = \prod_{k=1}^{\infty} (1 - x^k)$ the right hand of equation (17) is the number of partitions of k into parts prime to 3 and so equals to the number of partitions of k in which each part occurs at most two times. Using Maple for the product in equation (17) one obtain the following series

$$1 + x + 2x^2 + \cdots + 9x^7 + 13x^8 + \cdots + 1225x^{30} + \cdots .$$

In terms of partitions this means for example that the number of partition of the number 7 which are prime to 3 is 9 because

$$7 = 5 + 2 = 5 + 1 + 1 = 4 + 2 + 1 = 4 + 1 + 1 + 1 = 2 + 2 + 2 + 1 = 2 + 2 + 1 + 1 + 1 = 2 + 1 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1.$$

The sequence of partitions numbers corresponds to the coefficients in the series therefore this sequence is,

$$1, 1, 2, \cdots, 9, 13, \cdots, 1225, \cdots,$$

this sequence coincides with the sequence with reference number A00726 in the On-Line encyclopedia of integer sequences [7].

Next using the second factorization equation (15) we will give a different proof for the following three theorems. one theorem is on Andrews' result [5] in connection with generating functions that exclude squares and the other two theorems are their generalizations [6]. The Andrews' result stated in [6] by the following theorem;

Theorem 3.1 *Let $P_1(n)$ be the number of partitions of n in which each k appears at most $k - 1$ times and let $P_2(n)$ be the number of partitions of n with no squares part. Then, $P_1(n) = P_2(n)$.*

Proof. The generating function for $P_1(n)$ is nothing but the parafermionic partition function of order k , i.e.,

$$\sum_{n=0}^{\infty} P_1(n) x^n = \prod_{k=1}^{\infty} (1 + x^k + x^{2k} + \cdots + x^{(k-1)k}) = \prod_{k=1}^{\infty} \frac{(1 - x^{k^2})}{(1 - x^k)}, \quad (18)$$

the right hand side is the generating function for the number of partitions in which no square parts are present and so equals to $\sum_{n=0}^{\infty} P_2(n) x^n$ and hence $P_1(n) = P_2(n)$.

The generalization considered in [6] is to exclude polygonal numbers or r -gons as parts where the general polygonal number or the k^{th} -gonal number is given by $p_k^n = \frac{1}{2}k[(n-2)k - (n-4)]$ setting $n = 4$ this equation gives square numbers k^2 and $n = 5$ gives the k^{th} -pentagonal numbers $\frac{1}{2}k(3k-1)$, etc.

Theorem 3.2 *Let $r \geq 2$ be a fixed integer. Let $P_3(n, r)$ be the number of partitions of n in which each k appears at most $(r-1)(k-1)$ times and let $P_4(n, r)$ be the number of partitions of n where no $2r$ -gons can be used as parts. Then, $P_3(n, r) = P_4(n, r)$.*

Proof. The generating function for $P_3(n, r)$ is the parafermionic partition function of order $(r-1)(k-1) + 1$ and so we have;

$$\sum_{n=0}^{\infty} P_3(n, r) x^n = \prod_{k=1}^{\infty} (1 + x^k + x^{2k} + \cdots + x^{k(r-1)(k-1)}) = \prod_{k=1}^{\infty} \frac{(1 - x^{k[(r-1)(k-1)+1]})}{(1 - x^k)}, \quad (19)$$

to complete the proof we simply note that the term $k[(r-1)(k-1) + 1]$ can be rewritten as $k[(r-1)k - (r-2)] = \frac{1}{2}k[(2r-2)k - (2r-4)]$ which is the $2r$ -gons so the right hand side of the

above equation generates partitions whose parts are free of $2r$ -gons and hence $P_3(n, r) = P_4(n, r)$. Note that if we set $r = 2$ we obtain the previous result in which no square parts are present in the partitions. The next results obtained in [6] is to exclude $2r + 1$ -gons parts from partitions and is stated as follows;

Theorem 3.3 *Let $P_5(n, r)$ be the number of partitions of n in which each the part $2k - 1$ ($k \geq 1$) appears at most $(2r - 1)(k - 1)$ times (and the frequency of even parts is unbounded). let $P_6(n, r)$ be the number of partitions of n in which no odd-subscribed $2r + 1$ -gons can be used as parts. Then, for all non-negative n $P_5(n, r) = P_6(n, r)$.*

Proof. The generating function $P_5(n, r)$ from its definition is a product of even bosonic partition $\prod_{k=1}^{\infty} \frac{1}{(1-x^{2k})}$ and the odd parafermionic partition function of order $[(2r - 1)(k - 1) + 1]$. Therefore, similarly to the above proof we have

$$\begin{aligned} \sum_{n=0}^{\infty} P_5(n, r) x^n &= \prod_{k=1}^{\infty} \left(\frac{1}{1-x^{2k}} \right) \left(1 + x^{2k-1} + x^{2(2k-1)} + \dots + x^{(2k-1)(2r-1)(k-1)} \right) \\ &= \prod_{k=1}^{\infty} \left(\frac{1}{1-x^{2k}} \right) \left(\frac{1 - x^{(2k-1)[(2r-1)(k-1)+1]}}{1 - x^{(2k-1)}} \right) \\ &= \prod_{k=1}^{\infty} \left(\frac{1 - x^{(2k-1)[(2r-1)(k-1)+1]}}{1 - x^k} \right), \end{aligned} \quad (20)$$

from the following algebraic identity

$$\begin{aligned} (2k - 1)[(2r - 1)(k - 1) + 1] &= (2k - 1)[(2r - 1)k - (2r - 2)] \\ &= \frac{(2r + 1 - 2)}{2} (2k - 1)^2 - \frac{(2r + 1 - 4)}{2} (2k - 1), \end{aligned} \quad (21)$$

so the term $(2k - 1)[(2r - 1)(k - 1) + 1]$ is nothing but the $2r + 1$ -gons by definition, therefore $P_5(n, r) = P_6(n, r)$.

4 GRADED PARA Fermionic PARTITION FUNCTIONS AND OTHER IDENTITIES

We have already seen in section two that the graded fermionic partition function $\Delta_F(\beta) = \text{Tr} [(-)^F \exp -\beta(H_F)] = \prod_{k=1}^{\infty} (1 - x^k)$ so grading is equivalent to changing the sign of x^k in the fermionic partition function. Similarly the graded bosonic partition function can be obtained by changing the sign of x^k in the bosonic partition function, so if we denote by $\Delta_B(\beta)$ the graded bosonic partition function then we have

$$\begin{aligned} \Delta_B(\beta) &= \prod_{k=1}^{\infty} \frac{1}{(1 + x^k)} \\ &= \prod_{k=1}^{\infty} \sum_{i=0}^{\infty} (-1)^i x^{ik}, \end{aligned} \quad (22)$$

therefore the right hand of the above equation splits into even terms with plus sign in front and the odd terms with a minus sign in front and hence in terms of operators the graded bosonic partition

function can be written as $\Delta_B(\beta) = \text{Tr} [(-1)^{N_B} \exp -\beta(H_B)]$, where the operator $(-1)^{N_B} = \pm 1$. Note that the latter operator comes naturally here in trying to obtain the graded bosonic partition function. It was introduced in [3] in connection with the notion of the bosonic index; it is $+1$ for Fock space states with an even number of bosonic creation operators and -1 for Fock space states with an odd number of bosonic creation operators. Recall from the last section that the parafermion partition function is a truncation of the bosonic partition function and so the graded parafermion partition function would be the truncation of the graded bosonic partition function. There are two truncations to consider depending on the order s of the parafermion, when it is even $(-1)^s = +1$ or when it is odd $(-1)^s = -1$. we shall denote these partition functions by $\Delta_s^\pm(\beta)$. Therefore using our physical intuition, the graded parafermionic partition function is obtained from parafermionic partition function simply by changing x^k to $-x^k$. Therefore the identity in equation (16) with x^k changed to $-x^k$ gives another mathematical identity,

$$\prod_{k=1}^{\infty} (1 - x^k + x^{2k} - \dots + (-1)^{s-1} x^{(s-1)k}) = \prod_{k=1}^{\infty} \frac{(1 + (-1)^{s-1} x^{sk})}{(1 + x^k)}. \quad (23)$$

Rewriting the above identity in operator form and considering separately s even and odd, we end up with the following two formulae,

$$\begin{aligned} \Delta_s^+(\beta) &= \text{Tr} [(-1)^s \exp -\beta(H_s)] \\ &= \text{Tr} [(-1)^{N_B} \exp -\beta(H_B)] \text{Tr} [(-1)^F \exp -\beta(H_F)], \quad s \text{ even} \end{aligned} \quad (24)$$

$$\begin{aligned} \Delta_s^-(\beta) &= \text{Tr} [(-1)^s \exp -\beta(H_s)] \\ &= \text{Tr} [(-1)^{N_B} \exp -\beta(H_B)] \text{Tr} [\exp -\beta(H_F)], \quad s \text{ odd} \end{aligned} \quad (25)$$

which are the identities obtained in [3]. Using partial supersymmetry and the Witten index in the even case, and partial supersymmetry and the notion of the bosonic index in the odd case. As a consequence of the identities given in equations (16) and (23), the ratio of the graded parafermionic partition function to the parafermionic partition function can be written as;

$$\begin{aligned} \frac{\Delta_s^+(\beta)}{Z_s(\beta)} &= \frac{\prod_{k=1}^{\infty} (1 - x^k + x^{2k} - \dots - x^{(s-1)k})}{\prod_{k=1}^{\infty} (1 + x^k + x^{2k} + \dots + x^{(s-1)k})} \\ &= \prod_{k=1}^{\infty} \frac{(1 - x^k)}{(1 + x^k)}, \quad s \text{ even} \end{aligned} \quad (26)$$

$$\begin{aligned} \frac{\Delta_s^-(\beta)}{Z_s(\beta)} &= \frac{\prod_{k=1}^{\infty} (1 - x^k + x^{2k} - \dots + x^{(s-1)k})}{\prod_{k=1}^{\infty} (1 + x^k + x^{2k} + \dots + x^{(s-1)k})} \\ &= \prod_{k=1}^{\infty} \frac{(1 - x^k)}{(1 + x^k)} \frac{(1 + x^{sk})}{(1 - x^{sk})}, \quad s \text{ odd.} \end{aligned} \quad (27)$$

Here, we would like to make some remarks about the above identities and their implications. The first identity shows that for all s even, the ratio of the graded parafermionic partition function to

the parafermionic partition function is always given by $\prod_{k=1}^{\infty} \frac{(1-x^k)}{(1+x^k)}$. This ratio is known to be equal to $\theta_4(0, x) = \sum_{n=-\infty}^{\infty} (-1)^n x^{n^2}$ via Gauss's identity. The second identity, in the case of s odd, does not give $\theta_4(0, x)$ but rather the ratio $\theta_4(0, x)/\theta_4(0, x^s)$ so this extra factor can be thought of as correction factor that one has to insert in $\frac{\Delta_s^+(\beta)}{Z_s(\beta)}$ to obtain the odd case. Now by combining the above identities we obtain the following identity,

$$\begin{aligned} \frac{\Delta_s^-(\beta)}{Z_s(\beta)} &= \frac{\Delta_2^+(\beta)}{Z_2(\beta)} \prod_{k=1}^{\infty} \frac{(1+x^{sk})}{(1-x^{sk})} \\ &= \frac{\theta_4(0, x)}{\theta_4(0, x^s)}, \quad s \text{ odd.} \end{aligned} \quad (28)$$

Now, let us consider the graded parafermion of order three. Its partition function is $\Delta_3^-(\beta) = \prod_{k=1}^{\infty} (1 - x^k + x^{2k}) = \prod_{k=1}^{\infty} \frac{(1+x^{3k})}{(1+x^k)}$, and as for the parafermion of order three the right hand side of this identity can be simplified to give

$$\begin{aligned} \Delta_3^-(\beta) &= \prod_{k=1}^{\infty} (1 - x^k + x^{2k}) \\ &= \prod_{k=1}^{\infty} \frac{(1+x^{3k})}{(1+x^k)} \\ &= \prod_{k=1}^{\infty} \frac{1}{(1+x^{3k-2})} \frac{1}{(1+x^{3k-1})}. \end{aligned} \quad (29)$$

The inverse of this partition function is nothing but the generating function of the number of partitions of k into distinct parts which are prime to 3. We will see below that the generating function of Euler on partitions without restrictions is obtained by inverting the graded fermionic partition function of order two. In 1926, I.Schur [9] proved the following theorem

Theorem 4.1 *the number of partitions of k into distinct parts which are prime to 3 is identical with the number of partitions of a k into parts congruent to 1 or 5 modulo 6, in addition both of these number of partitions are equal to the number of partitions of k of the form $b_1 + b_2 + \dots + b_l$ such that $b_i - b_{i+1} \geq 3$ with strict inequality if b_i is a multiple of 3.*

Therefore as a result we obtain the following identity,

$$\frac{1}{\Delta_3^-(\beta)} = \prod_{k=1}^{\infty} \frac{1}{(1-x^{6k-5})} \frac{1}{(1-x^{6k-1})}, \quad (30)$$

so we learn that the inverted graded parafermionic partition function of order 3 coincides with the generating function for the numbers of partitions in the Schur's theorem. In the following we would like to make a connection between the graded parafermion of even order, say $s = 2l$, l any positive integer and the graded parafermion of order l . Since the term $(1 + (-1)^{2l-1} x^{2lk})$ in the expression of the graded parafermionic partition function factorizes and hence the expression

relating the inverted parafermionic partition functions of order $2l$ to that of order l is,

$$\begin{aligned}\frac{1}{\Delta_{2l}^+(\beta)} &= \prod_{k=1}^{\infty} \frac{(1+x^k)}{(1+(-1)^{2l-1}x^{2lk})} \\ &= \prod_{k=1}^{\infty} \frac{(1+x^k)}{(1+(-1)^{l-1}x^{lk})} \frac{1}{(1+(-1)^lx^{lk})}.\end{aligned}\quad (31)$$

The parafermionic partition function of order one is the Witten index

$$\Delta_1^-(\beta) = \text{Tr} [(-)^F \exp -\beta(H_B + H_F)] = 1,$$

so both the the graded and the inverted parafermionic partition functions are equal to 1. Therefore the inverted graded parafermionic partition function of order two is,

$$\begin{aligned}\frac{1}{\Delta_2^+(\beta)} &= \prod_{k=1}^{\infty} \frac{(1+x^k)}{(1-x^{2k})} \\ &= 1 \prod_{k=1}^{\infty} \frac{1}{(1-x^k)},\end{aligned}\quad (32)$$

which is exactly the Euler generating function for the unrestricted partitions $p(k)$ of a number k that we mentioned above. The order 4 inverted parafermionic partition function should be related to the Euler generating function,

$$\begin{aligned}\frac{1}{\Delta_4^+(\beta)} &= \prod_{k=1}^{\infty} \frac{(1+x^k)}{(1-x^{4k})} \\ &= \prod_{k=1}^{\infty} \frac{1}{(1-x^k)} \frac{1}{(1+x^{2k})} \\ &= \prod_{k=1}^{\infty} \frac{(1+x^{2k-1})}{(1-x^{2k})},\end{aligned}\quad (33)$$

where the right hand side of this equation gives the number of partitions of k in which each even part occurs with even multiplicity and there is no restriction on the odd distinct parts. the next generating function we look for is the one connected with the Schur theorem and it is the inverted sixth order parafermionic partition function,

$$\begin{aligned}\frac{1}{\Delta_6^+(\beta)} &= \prod_{k=1}^{\infty} \frac{(1+x^k)}{(1-x^{6k})} \\ &= \prod_{k=1}^{\infty} \frac{(1+x^{3k-1})(1+x^{3k-2})}{(1-x^{3k})}.\end{aligned}\quad (34)$$

This gives the number of partitions of k in which the distinct parts are prime to 3 and the unrestricted parts contain 3 and its multiples. Using Maple one could generate sequences of partitions from our formulae depending on s . When $s = 3$ we have,

$$1 + x + x^2 + \cdots + 3x^7 + 3x^8 + 3x^9 + 4x^{10} + \cdots$$

so for example the number of distinct partitions of the number 10 in which the parts are prime to 3 is 4 (which we denote by $a(10) = 4$) because $10 = 8 + 2 = 7 + 2 + 1 = 5 + 4 + 1$. The corresponding sequence of partitions is,

$$1, 1, 1, 1, 1, 2, 2, 3, 3, 3, 4, 5, 6, 7, 8, 9, 10, 12, \dots,$$

this corresponds to the sequence A003105 on the on-Line Encyclopedia of integer sequences [7]. The next sequence of partitions of the number k would be that which corresponds to $s = 4$, which is,

$$1, 1, 1, 2, 3, 4, 5, 7, 10, 13, 16, 21, \dots$$

this is also in [7] and corresponds to the sequence A006950. It was pointed out by Sellers that the number of partitions for this sequence that we mentioned above for $s = 4$ it is also the number of partitions of k into parts not congruent to 2 mod 4. so for example number of partition of 7 equals 7, $a(7) = 7$ because

$$7 = 5 + 1 + 1 = 4 + 3 = 4 + 1 + 1 + 1 = 3 + 3 + 1 = 3 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1.$$

The fifth and the sixth orders from our formulae for the inverted parafermionic partition function give the following sequences of partition of the number k ,

$$1, 1, 1, 2, 2, 2, 3, 4, 4, 6, 7, 8, 10, 16, 19, 22, 26, 30, 35, 41, \dots$$

for the fifth order and

$$1, 1, 1, 2, 2, 3, 5, 6, 7, 10, 12, 15, 25, 30, 39, 46, \dots$$

for the sixth order respectively. These two sequences are new sequences and now they appear on the on-Line Encyclopedia of integer sequences [7] with references A096938 and A096981. In the fifth order inverted parafermionic partition function, the number 5 and its multiples are not present in the number of partitions of k into distinct parts, for example $8 = 7 + 1 = 6 + 2 = 4 + 3 + 1$ so $a(8) = 4$ also $9 = 8 + 1 = 7 + 2 = 4 + 3 + 2$ then $a(9) = 4$. For the sixth order sequence the number of partitions of the number 11 for example, is 15, $a(11) = 15$. Because

$$11 = 10 + 1 = 8 + 2 + 1 = 7 + 4 = 5 + 4 + 2 = 9 + 2 = 8 + 3 = 7 + 3 + 1 = 6 + 5 = 6 + 4 + 1 = 6 + 3 + 2 = 5 + 3 + 2 + 1 = 4 + 3 + 3 + 1 = 3 + 3 + 3 + 2.$$

Therefore, we see that our generating function gives known sequences for partitions of numbers and as well as new ones. We would like to make a comment about our generating function, for example in the third order see the left hand side of equation (30) this was given in the sequence A00305. At the fourth order, however, a formula like ours was not given in the sequence A006950 see [7].

Finally we mention another new sequence that we have found in this work and which follows from the identity in equation (28). Setting $s = 3$ then we have

$$\frac{Z_3(\beta)}{\Delta_3^-(\beta)} = \frac{\theta_4(0, x^3)}{\theta_4(0, x)},$$

and using Maple the sequence is,

$$1, 2, 4, 6, 10, 16, 24, 36, 52, 74, 104, 144, 198, 268, 360, \dots, 48672, 59122, \dots$$

This corresponds to the number of partitions of $2k$ prime to 3 with all odd parts occurring with even multiplicities. There is no restriction on the even parts e.g $a(8) = 10$ because

$$\begin{aligned} 8 &= 4 + 4 = 4 + 2 + 2 = 2 + 2 + 2 + 2 = 2 + 2 + 2 + 1 + 1 = 2 + 2 + 1 + 1 + 1 + 1 = \\ &2 + 1 + 1 + 1 + 1 + 1 + 1 = 4 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 + 1 + 1. \end{aligned}$$

This new sequence has reference number A098151 in the on-Line Encyclopedia of integer sequences [7]. From the identity given by equation (27) for s even one has,

$$\frac{Z_s(\beta)}{\Delta_s^+(\beta)} = \frac{1}{\theta_4(0, x)},$$

this generates a sequence which coincides with the sequence A0151128 in [7], using Maple we obtain

$$1, 2, 8, 14, 24, 40, 64, 100, 154, \dots, 9904, 13288, \dots,$$

which is the number of partitions of $2k$ with all odd parts occurring with even multiplicities. There is no restriction on the even parts.

5 CONCLUSIONS

In this paper partial supersymmetry was used to derive Euler theorem from a fermionic partition function and from the parafermionic partition function a theorem in the theory of partitions, which says that the number of partitions of k in which no parts appear more than $s - 1$ times equals the number of partitions of k such that no parts is divisible by s . In the last section we obtained the expression for the graded parafermionic partition function and we saw that the inverted parafermionic partition function generates partition of numbers with given restrictions. The generating function we have obtained is general in the sense that when the order of the graded parafermion is two our generating function coincides with that of the Euler generating function of partitions without restrictions. We have also shown that these generating functions are related to each other, the order $2l$ is expressed in terms of order l where l is any positive integer. Also in the last section we have obtained new sequences of partitions [7]. The theory used here may be called additive quantum theory as the bosonic partition function is the Euler generating function for the unrestricted partitions. The Euler theorem equates the number of distinct partitions with the number of unrestricted odd partitions. The generating function proof of this theorem is simple in number theory and this is also the case using partial supersymmetry. One interpretation we give is that the fermionic partition function is the odd part of the bosonic partition function. One may also look at it differently and write $Z_f(\beta) = \Delta_F(2\beta)/\Delta_F(\beta)$ from which we learn that $\Delta_F(2\beta) = Z_f(\beta)\Delta_F(\beta)$ so mixing the fermionic system with the graded fermionic system at thermal equilibrium at one temperature is the same as a graded parafermionic system with different temperature. This also happens in the case of quantum field theory with a logarithmic spectrum [3] in which the term duality was used to characterize the identities among arithmetic quantum theories. In our case the latter formula generalizes naturally by simply using equation (16) or equations (26) and (27) which read, $\Delta_F(s\beta) = Z_s(\beta)\Delta_F(\beta)$ for both s even and odd, this identity will be a trivial identity when s goes to infinity as both bosonic and graded fermionic partition

functions cancel each other and we recover the Witten index which is a complete cancellation between boson and fermions. An other way to look at the equations (26) and (27) is that in the former equation the ratio of the graded parafermionic partition function to the parafermionic partition function when s is even is always given by $\theta_4(0, x)$. When s is odd, however, it is given by the ratio $\theta_4(0, x)/\theta_4(0, x^s)$. Finally, the new sequence A096981 for the sixth order that I have obtained is also equivalent to the number of partitions of k into parts congruent to 0,1,3,5 mod 6 see sequence A096981 in [7] for details on the sequence and its connection with other sequences.

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